# CONTINUOUS AND DISCRETE FOURIER SPECTRA OF APERIODIC SEQUENCES FOR SOUND MODELING

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**ABSTRACT**

The Fourier analysis of aperiodic ordered time structures related with number eight is considered. Recursion relations for the Fourier amplitudes are obtained for a sequence with discrete spectrum. The continuous spectrum of a different type of sequence is also studied . By increasing the number of points in the time axis dynamic spectra can be obtained and used for sound synthesis.

## 1 Introduction.

Selfsimilar aperiodic distributions of impulses in the time axis and their Fourier analysis has been used in recent years for sound synthesis [1,2]. The basic structures are deterministic aperiodic ordered sequences described with Lindenmayer systems .

Discrete and continuous spectra appear in the Fourier analysis. The presence of discrete components depends on the nature of the scaling factor in the sequences. A Pisot -Vijayaraghavan (PV) number is an algebraic integer greater than one with all its conjugates (the remaining roots of its minimal polynomial) strictly less than one in absolute value. If the scaling factor is a PV number then there is, up to possible extintions, a discrete component [3]. This type of spectrum is inharmonic due to the lack of translational symmetry. Temporal sequences with PV numbers as scaling factors related with numbers five and seven have been discussed in [1] and sequences with PV inflation factors related with number nine appear in [4,6].

In this work several sequences related with number eight are analysed. One of them has been the basis for the derivation of a species of octagonal aperiodic tilings of the plane [5]. Tiling growth is described in terms of deterministic or stochastic L systems, and the word sequences can be translated into harmonic fields progressions [6]. Sequences having non PV numbers as inflation factors show no discrete part in the Fourier transform. Evolutions in time for the phases and amplitudes are the result of an increment in the number of time impulses.

## 2 Fourier Analysis of an aperiodic ordered structure with discrete spectrum.

A D0L system consists in an alphabet, a set of production rules for the allowed words, and an axiom or starting symbol. Consider the alphabet with the letters L and S, the production rules h:

$$L \rightarrow LLS \qquad\qquad S \rightarrow L$$

and the axiom L. The language consists in the words $h^N[L]$:

L, LLS, LLSLLSL, LLSLLSLLLSLLSLLLS,....

A temporal sequence can be associated to the words by taking two segments with

(length of L)/(length of S)= $\theta$= $1+2\cos[\pi/4]$.

With this choice the sequence is selfsimilar with scaling factor $\theta$ which is a PV number with minimal polynomial

$x^2-2x-1$

The distribution of impulses following the sequence LS is represented by

$$\rho(t)= \sum_k \delta(t- t[k])$$

where the Dirac delta-function $\delta(x)$ has the properties: $\delta(x) = 0$ unless $x=0$, $\delta(0) = \infty$, and

$t[m]=2t[m-1]+t[m-2]$
$t[0]=L$, $t[1]=2L+S$

Its Fourier transform is

$\lim_{N\to\infty}((1/c[N])\Sigma_k \exp(2\pi i Y t[k]))$

where c[N] is the number of letters of the word $h^N[L]$. In general the Fourier spectrum of the sequence is a sum of discrete, continuous (it is the primitive of its derivative) and singularly continuous (its derivative vanishes almost everywhere) components [7,8]. It is possible to derive recursion relations for the Fourier amplitudes in the LS-system for the discrete part of the spectrum:

$f[n]=1+(\exp(2\pi i Y t[n-1]))f[n-1]+$
$(\exp(4\pi i Y t[n-1]))f[n-2],$

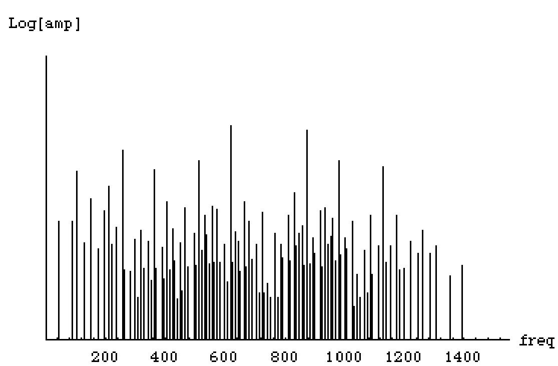
$f[0]=1,$
$f[1]=1+\exp(2\pi i Y L)+\exp(2\pi i Y(2L+S))$

where

$Y= p\omega_1 +q\omega_2$

$\omega_1=\theta/(\theta+1)$
$\omega_2 =1/(\theta+1)$
$p,q\in Z$

By increasing the level of iteration N, dynamic spectra are obtained. The module of the complex number amp=f[n]/c[n] stabilizes after eight iterations (see Fig. 1)



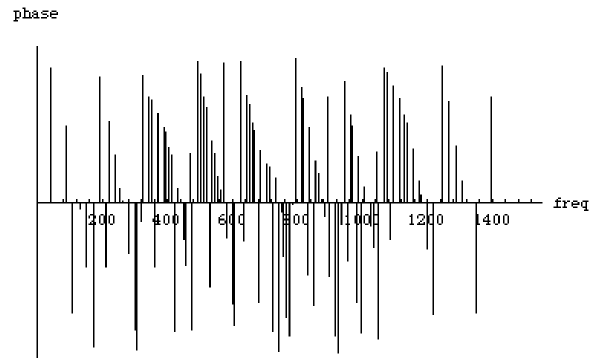
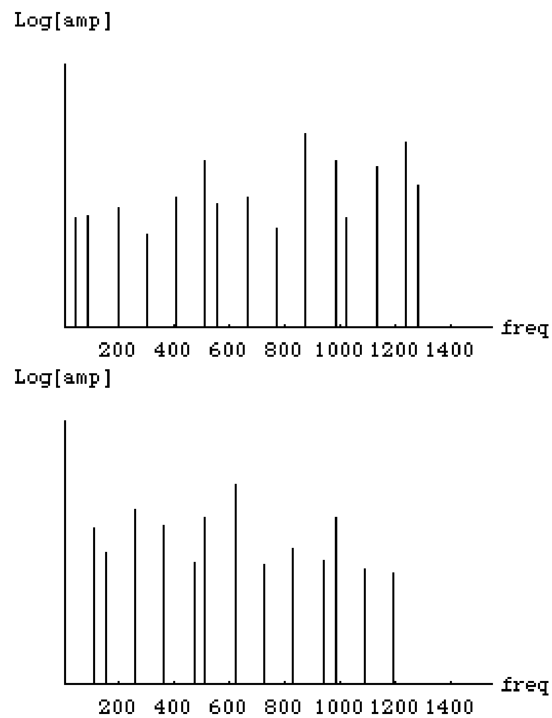




Figure 1: Discrete Spectrum for the LS-System for N=8.

Sets of partials separated by intervals of lengths L and S in the frequency space can be obtained from the spectrum. In Fig.2 four subsets of such partials are shown



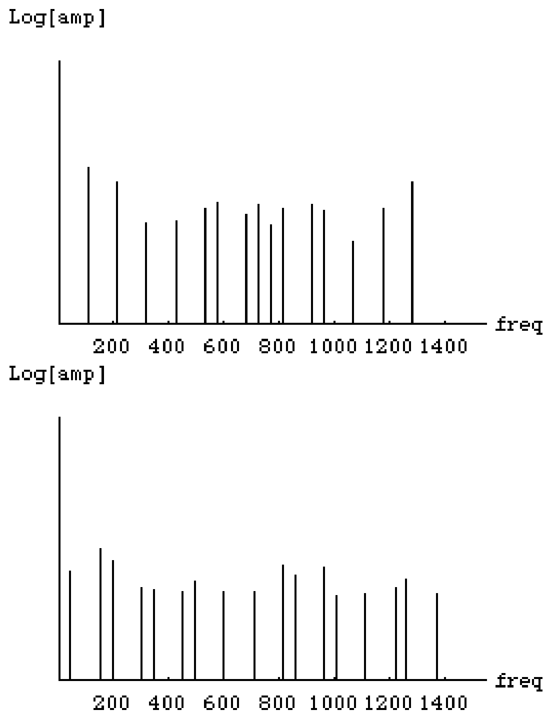



Figure 2: Sets of selected partials from the LS-spectra with frequencies separated by distances L or S.

## 3 Temporal sequences with continuous spectrum .

In [5] selfsimilar aperiodic tilings of the plane with eightfold symmetry have been introduced. For the deterministic cases three types of scaling factors are possible. The prototiles are allways triangles and the substitution rules for their edges in one of the species are

$H_0 \rightarrow I_1 \qquad H_1 \rightarrow I_0$

$I_0 \rightarrow H_1\, J_0 \qquad I_1 \rightarrow J_1\, H_0$

$J_0 \rightarrow K\, I_0 \qquad J_1 \rightarrow I_1\, K$

$K \rightarrow J_0\, J_1$

with (length of I)/(length of H)= sin[π/4]/ sin[π/8]; (length of J)/(length of H)= sin[3π/8]/ sin[π/8]; (length of K)/(length of H)= sin[π/2]/ sin[π/8].
In this case two different sequences are obtained. If the axiom is H then odd iterations give strings of symbols of type I and K. Strings with only H,J are obtained when even iterations are considered. In Fig.3 a small part of the frequency domain representation for the sixth iteration is shown. The frequencies are scaled by a factor 1/150. Two steps in the spectral evolution for the odd iterations are represented in Figs.4 and 5 where we can see also how some peaks in the spectrum appear with narrower bandwidth when the iteration level is increased.

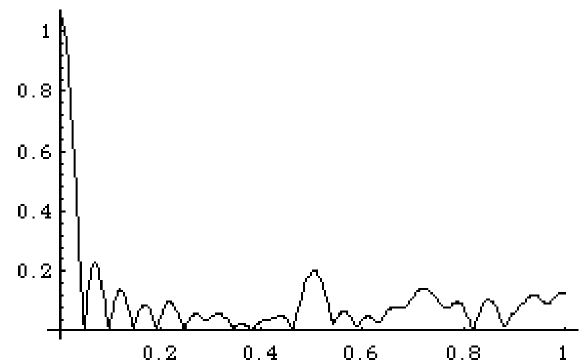



Fig.-3 Continuous Spectrum for the HJ-system. Level of iteration n=6

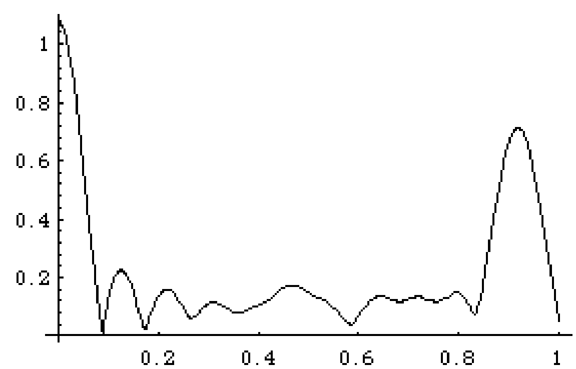



Fig.-4 Continuous Spectrum for the IK-system. Level of iteration n=5.

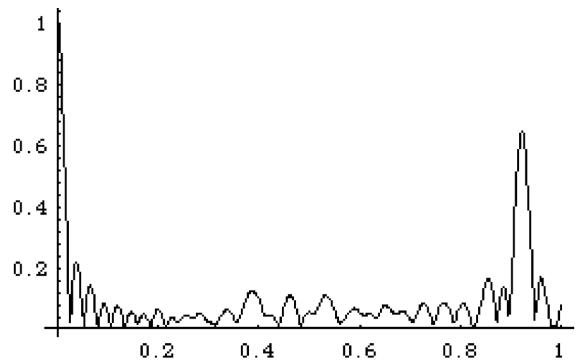



Fig.-5 Continuous Spectrum for the IK-system. Level of iteration n=7.

## 4 Concluding remarks.

The point sets forming the temporal geometries considered in this work belong to a more general class known as Delone sets studied in crystallography [9]. They have two fundamental properties: the points can not be placed arbitrarily close together and the point distribution is uniform enough. Whether or not they are repetitive [9] or if the atlas of local configurations is finite are questions necessary to understand in order to clarify general characteristics of their Fourier spectra.

Several examples of Delone sets with discrete and continuous spectra have been analysed. The basic geometric structures have a deterministic nature. Nondeterministic tilings of the plane can be described in terms of stochastic bracketed Lindenmayer systems [5]. As in section 3 one can obtain 1D substitutions from the tile edges but in that case they have the property that they are not deterministic. In the Fourier transform they show peaks with a continuous background. When several substitution or production rules are considered, each one with a certain probability of occurrence, a great variety of dynamic spectra can be derived for experimentation in sound design.

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