ISOMORPHISMS

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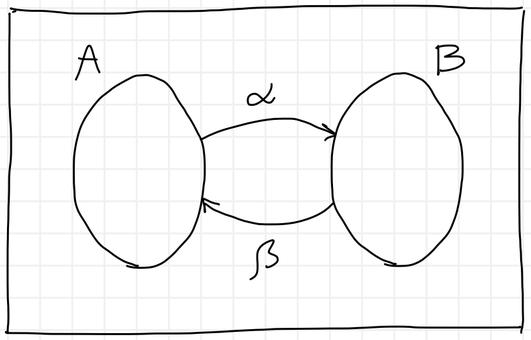
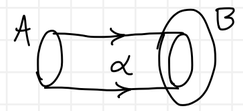
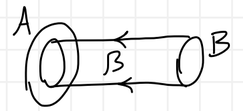
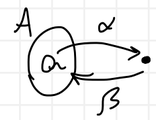
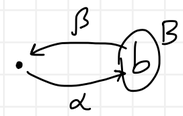
Isomorphic Mapping



$A \subseteq \mathcal{U}$, $B \subseteq \mathcal{U}$ } domains (unary predicates)

$\alpha : \mathcal{U} \to \mathcal{U}$, $\beta : \mathcal{U} \to \mathcal{U}$ } conversions (unary functions)

conditions:

- $\boxed{\alpha A}$ $\forall a \in A.\ \alpha(a) \in B$ — α maps A to B  $\alpha(A) \subseteq B$
- $\boxed{\beta B}$ $\forall b \in B.\ \beta(b) \in A$ — β maps B to A  $\beta(B) \subseteq A$
- $\boxed{\beta\alpha}$ $\forall a \in A.\ \beta(\alpha(a)) = a$ — β is left inverse of α over A 
- $\boxed{\alpha\beta}$ $\forall b \in B.\ \alpha(\beta(b)) = b$ — α is left inverse of β over B 

$A \underset{\beta}{\overset{\alpha}{\leftrightarrows}} B \triangleq \alpha A \wedge \beta B \wedge \beta\alpha \wedge \alpha\beta$ — α and β are mutually inverse isomorphisms between A and B

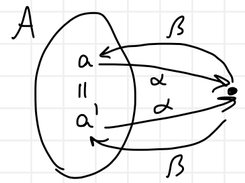
⊢ [αi] $\forall a, a' \in A.\ \alpha(a) = \alpha(a') \Rightarrow a = a'$ — α injective on A

$a \in A$ $\quad \alpha(a) = \alpha(a')$ $\quad a' \in A$

$\beta\alpha$: $\beta(\alpha(a)) = \beta(\alpha(a'))$: $\beta\alpha$

$\beta(\alpha(a)) = a$ $\quad \beta(\alpha(a')) = a'$

QED



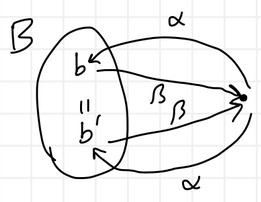
⊢ [βi] $\forall b, b' \in B.\ \beta(b) = \beta(b') \Rightarrow b = b'$ — β injective on B

$b \in B$ $\quad \beta(b) = \beta(b')$ $\quad b' \in B$

$\alpha\beta$: $\alpha(\beta(b)) = \alpha(\beta(b'))$: $\alpha\beta$

$\alpha(\beta(b)) = b$ $\quad \alpha(\beta(b')) = b'$

QED



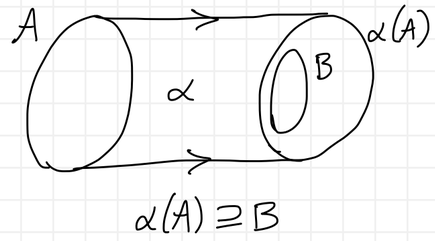
⊢ [αs] $\forall b \in B.\ \exists a \in A.\ b = \alpha(a)$ — α surjective on B from A

$b \in B \xrightarrow{\alpha\beta} \alpha(\beta(b)) = b$

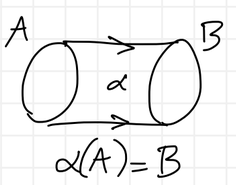
βb : $a \triangleq \beta(b) \to \alpha(a)$

$a \in A$

QED



αA



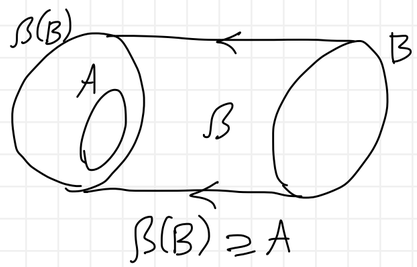
⊢ [βs] $\forall a \in A.\ \exists b \in B.\ a = \beta(b)$ — β surjective on A from B

$a \in A \xrightarrow{A\iota} \beta(\alpha(a)) = a$

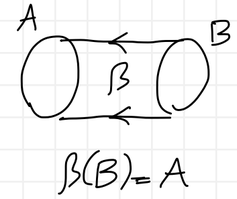
αA : $b \triangleq \alpha(a) \to \beta(b)$

$b \in B$

QED



βB



Guards

conditions
- GA: $\gamma_A = \mathcal{U}$ — A well-defined everywhere
- GB: $\gamma_B = \mathcal{U}$ — B well-defined everywhere
- Gα: $\gamma_\alpha \supseteq A$ — α well-defined at least over A
- Gβ: $\gamma_\beta \supseteq B$ — β well-defined at least over B

$G[A \underset{\beta}{\overset{\alpha}{\leftrightarrows}} B] \triangleq GA \wedge GB \wedge G\alpha \wedge G\beta$ — the constituents of $A \underset{\beta}{\overset{\alpha}{\leftrightarrows}} B$ satisfy the guard conditions

$\vdash \boxed{\surd_\alpha A}\ \omega_{\alpha A}(a)$

$\omega_{\alpha A}(a) = [\cancel{\gamma_A(a)} \wedge [a \in A \Rightarrow \cancel{\gamma_\alpha(a)} \wedge \cancel{\gamma_B(\alpha(a))}]]$

($\gamma_A(a)$: GA; $a \in A \Rightarrow \gamma_\alpha(a)$: G$\alpha$; $\gamma_B(\alpha(a))$: GB)

QED

$\vdash \boxed{\surd_\beta B}\ \omega_{\beta B}(b)$

$\omega_{\beta B}(b) = [\cancel{\gamma_B(b)} \wedge [b \in B \Rightarrow \cancel{\gamma_\beta(b)} \wedge \cancel{\gamma_A(\beta(b))}]]$

($\gamma_B(b)$: GB; $b \in B \Rightarrow \gamma_\beta(b)$: G$\beta$; $\gamma_A(\beta(b))$: GA)

QED

$\vdash \boxed{\surd_{\beta\alpha}}\ \omega_{\beta\alpha}(a)$

$\omega_{\beta\alpha}(a) = [\cancel{\gamma_A(a)} \wedge [a \in A \Rightarrow \cancel{\gamma_\alpha(a)} \wedge \cancel{\gamma_\beta(\alpha(a))}]]$

($\gamma_A(a)$: GA; $a \in A \Rightarrow \gamma_\alpha(a)$: G$\alpha$; αA: $\alpha(a) \in B$; $\gamma_\beta(\alpha(a))$: Gβ)

QED

$\vdash \boxed{\surd_{\alpha\beta}}\ \omega_{\alpha\beta}(b)$

$\omega_{\alpha\beta}(b) = [\cancel{\gamma_B(b)} \wedge [b \in B \Rightarrow \cancel{\gamma_\beta(b)} \wedge \cancel{\gamma_\alpha(\beta(b))}]]$

($\gamma_B(b)$: GB; $b \in B \Rightarrow \gamma_\beta(b)$: G$\beta$; βB: $B(b) \in A$; $\gamma_\alpha(\beta(b))$: Gα)

QED

Generalization to Tuples

$A \subseteq \mathcal{U}^n \qquad B \subseteq \mathcal{U}^m \qquad \alpha : \mathcal{U}^n \to \mathcal{U}^m \qquad \beta : \mathcal{U}^m \to \mathcal{U}^n$

everything works the same as in the unary case

Variant : Unconditional Theorems

$\boxed{\beta\alpha'}$ $\forall a.\ \beta(\alpha(a)) = a$ — holds for $a \notin A$ too

$\boxed{\alpha\beta'}$ $\forall b.\ \alpha(\beta(b)) = b$ — holds for $b \notin B$ too

⊢ $\boxed{\alpha i'}$ $\forall a, a'.\ \alpha(a) = \alpha(a') \Rightarrow a = a'$ — holds for $a \notin A$ or $a' \notin A$ too

$\alpha(a) = \alpha(a')$

$a \overset{\beta\alpha'}{=} \beta(\alpha(a)) = \beta(\alpha(a')) \overset{\beta\alpha'}{=} a'$

QED

⊢ $\boxed{\beta i'}$ $\forall b, b'.\ \beta(b) = \beta(b') \Rightarrow b = b'$ — holds for $b \notin B$ or $b' \notin B$ too

$\beta(b) = \beta(b')$

$b \overset{\alpha\beta'}{=} \alpha(\beta(b)) = \alpha(\beta(b')) \overset{\alpha\beta'}{=} b'$

QED

making also αA and βB unconditional seems unnecessary : just have $A = B = \mathcal{U}$ instead